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COMPARATIVE ANALYSIS OF SOLUTION METHODS TO H_∞ SYNTHESIS PROBLEM ACROSS DIFFERENT SYSTEM TYPES

Yusuf Çağrı Öğüt*
Hacettepe University
Ankara, Türkiye

Burak Kürkçü†
Santa Clara University
California, United States

Mehmet Önder Efe‡
Hacettepe University
Ankara, Türkiye

ABSTRACT

In this paper, comparative analysis of solution methods, 2-Riccati and Linear Matrix Inequalities (LMI), to $\mathcal{H}_\infty$ synthesis problem are conducted across different system types. The systems are selected based on the assumptions underlying each method and their significance in the control theory literature. The analysis is performed with varying parameter settings, and the results are evaluated in terms of the iteration process, the feasibility of the synthesized controllers, and robustness metrics. It has been observed that the LMI solution is generally more flexible with the restrictions. Nevertheless, for proper systems, its computational complexity is inferior. Another important conclusion is that numerical conditions and regularization are crucial for both solution methods.

INTRODUCTION

Perturbation, disturbance and modeling errors lead to a decrease in system stability and performance. With $\mathcal{H}_\infty$ synthesis, a controller which is stabilizing, which means the worst case gain is smaller than a preset value, and ensures guaranteed performance can be designed for this problem. However, some assumptions regarding the solutions to $\mathcal{H}_\infty$ synthesis impose some restrictions on the systems. These restrictions are well studied in [5; 3]. Even though broad analysis are done on matrix level, there is not much effort put into the problem in systems perspective.

This paper focuses mainly on solution methods to suboptimal $\mathcal{H}_\infty$ controller synthesis problem across different system types. Minimum necessary background is given. A comprehensive literature survey is done. Systems, which are potentially problematic for the solutions and important in control theory literature, are modeled and used in analysis. Comparative analysis are carried out regarding the solution performances.

Literature Review

It can be stated that Linear Quadratic Regulator (LQR) is the start of modern control systems engineering. It simply uses a cost function that consists of penalties for error and control effort. The major drawback of LQR was that it assumed perfect-state measurements, which is not realistic. This drawback led to development of Linear Quadratic Gaussian (LQG). LQG extended LQR with

*Researcher, Email: yusufcagriogut@gmail.com
†Researcher, Email: bkurkcu@scu.edu
‡Researcher, Email: onderefe@gmail.com

a Kalman filter in purpose of assuming the state measurements can be faulty and they should be corrected. LQG assumes white-noise uncertainties, nevertheless parameter uncertainty is not always so. It was also still a problem to inspect the control problem as a quadratic equation. This approach worked well for performance criteria, nevertheless, stabilization was still an issue. In [6], Doyle et al. showed that there can not be any guaranteed margins for it.
Robust control emerged in 1980's. In robust control, mostly static controllers are used rather than the adaptive controllers, which tracks the variations in measurements. In contrast to the simple robust stability augmentation such as high gain feedback controllers, robust control techniques avoid high gains for enhanced closed-loop stability. Robust control methods include Sliding Mode Controllers, μ-synthesis, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ synthesis.
$\mathcal{H}_\infty$ synthesis emerged in the 1980's [1]. In [1], problem formulation is presented. In [2] a solution method for $\mathcal{H}_\infty$ synthesis, which only requires solving two Riccati equations is proposed with a comprehensive background and special problems, but proof to the solution is not given. Restrictions emerging due to the assumptions are also discussed. Even though some proposed modifications [2] for the assumptions to the 2-Riccati solution work well for transmission zeros at infinity [7; 8; 9; 10], other assumptions are still an issue [11].
In [3], a new solution method, which uses Linear Matrix Inequalities (LMI) rather than the algebraic Riccati equations is proposed. This approach is superior in sense of working better with limitations such as invariant/transmission zeros.
Significant amount of effort is given for the matrix level analysis of $\mathcal{H}_\infty$ synthesis. In [5; 3] the assumptions of the 2-Riccati solution are broadly investigated; relaxing assumptions regarding the normalization of D_{12} and D_{21} matrices, removing the $D_{22} = 0$ assumption, allowing non-stabilizable and non-detectable modes, rank deficiency of D_{12} and D_{21} matrices are studied. Nevertheless, analysis from the system perspective for this problem is not well studied. In [4], 2-Riccati and LMI solutions are compared for systems with invariant/transmission zeros and feed-through matrix with rank deficiency for second order mass-spring-damper system and active suspension system. As the author depicts a broader study is needed.

Background

System Formulation for $\mathcal{H}_\infty$ Synthesis

The system formulation for the system in linear fractional transform (LFT) form (1) is given in (1), (2) and (3).

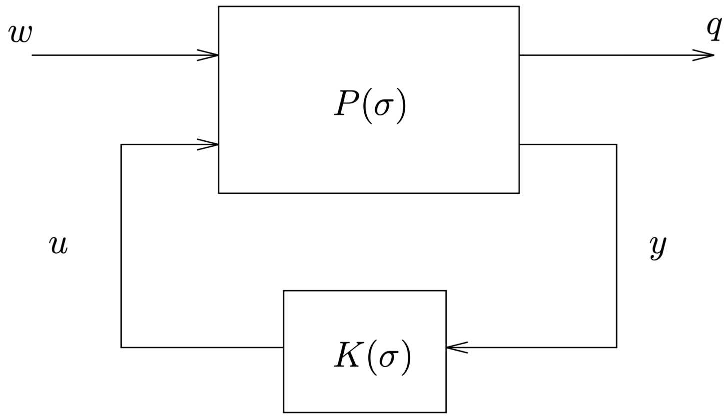

Figure 1: System Block Diagram in LFT Form

$$\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix} \tag{1}$$

here z denotes the controlled output, y represents the measurement output, w stands for the disturbance input and u is the control input. Plant, P, is given in (2) [5].

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{2}$$

The minimal realization of the plant [5] P is given in (3) [5].

$$P = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (\sigma I - A)^{-1}(B_1, B_2) \tag{3}$$

(3) corresponds to (4) in open form [5].

$$\begin{aligned} \dot{q}(t) &= Aq(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 q(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 q(t) + D_{21} w(t) + D_{22} u(t) \end{aligned} \tag{4}$$

where $q(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $z(t) \in \mathbb{R}^{n_z}$ is the controlled output vector, $y(t) \in \mathbb{R}^{n_y}$ is the measured output vector, $A \in \mathbb{R}^{n \times n}$ is the system matrix, $B_1 \in \mathbb{R}^{n \times n_w}$ is the disturbance input matrix, $B_2 \in \mathbb{R}^{n \times n_u}$ is the control input matrix, $C_1 \in \mathbb{R}^{n_z \times n}$ is the controlled output matrix, $C_2 \in \mathbb{R}^{n_y \times n}$ is the measured output matrix, $D_{11} \in \mathbb{R}^{n_z \times n_w}$ is the matrix that maps the disturbance input to the controlled output, $D_{12} \in \mathbb{R}^{n_z \times n_u}$ is the mapping between the controlled input to the controlled output, $D_{21} \in \mathbb{R}^{n_y \times n_w}$ is the matrix from the disturbance input to the measurement output, $D_{22} \in \mathbb{R}^{n_y \times n_u}$ represents the the matrix linking the controlled input to the measurement output, n is the state dimension, n_y is the dimension of system measurements, n_u is the dimension of control inputs, n_z is the dimension of controlled outputs, and n_w is the dimension of the disturbance inputs [5].

$\mathcal{H}_\infty$ Norm

Given a stable LTI system in (4), $\mathcal{H}_\infty$ norm is formulated as in (5) [5].

$$\|G(s)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}\left(G(j\omega)\right) \tag{5}$$

where $\bar{\sigma}(G(j\omega))$ denotes the greatest singular value of $G(j\omega)$, and sup is the supremum (smallest upper bound) [5]. This value corresponds to the largest singular value on Bode magnitude plot.

It would be unpractical to calculate transfer function's gain in each frequency or draw Bode plot and check the largest gain. Considering that this is needed in real-time applications as it is used in controllers, rather than calculating $\mathcal{H}_\infty$ norm, depicting a value and checking if there exists a proper controller for that value is a better alternative. This approach is used in different solution methods to $\mathcal{H}_\infty$ synthesis.

A classical example for a MIMO system is given in (6).

$$G = \begin{bmatrix} 0 & \dfrac{3s}{s^2 + s + 10} \\ \dfrac{s+1}{s+5} & \dfrac{2}{s+6} \end{bmatrix} \tag{6}$$

$\mathcal{H}_\infty$ norm can be calculated by sweeping all frequencies. Bode gain plot can be seen in Figure 2.

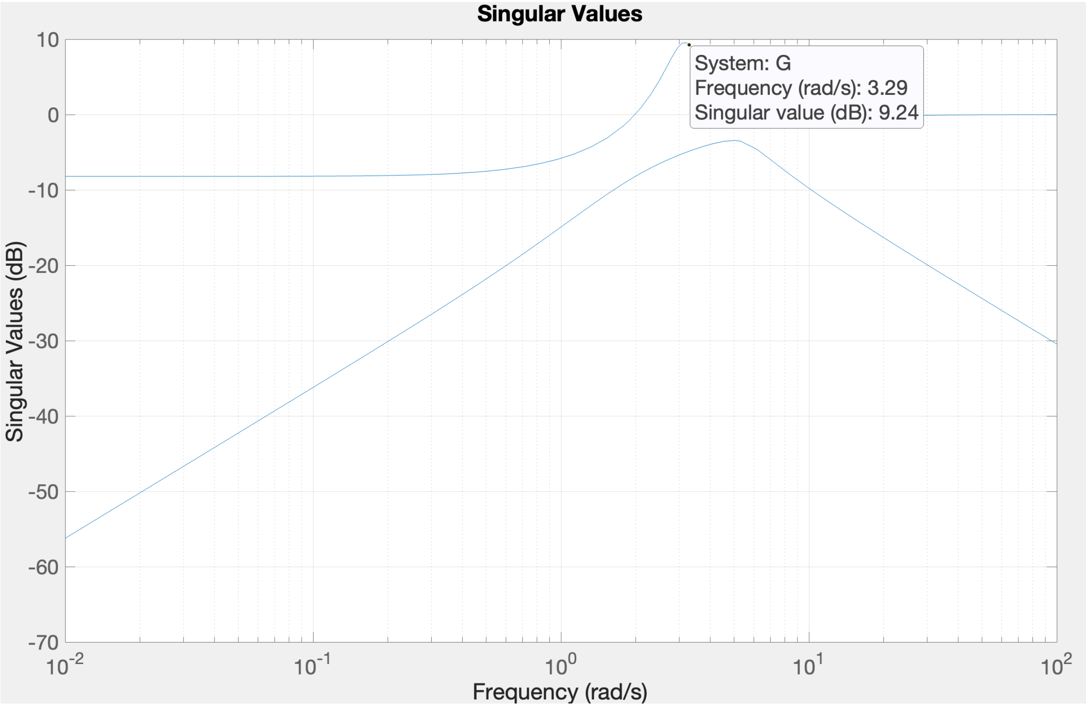

Figure 2: Gain Plot for the System in (6)
for Both Singular Values

It can be seen that the peak gain occurs at 3.29 rad/sec and the value is 9.24 dB.

Suboptimal $\mathcal{H}_\infty$ Control Problem

Finding the optimal $\mathcal{H}_\infty$ controller is very hard, even not desired generally [5]. Suboptimal $\mathcal{H}_\infty$ controller synthesis problem is concerned with synthesizing gain matrix $K(s)$ so that mapping $T_{zw}(s)$ from $w(t)$ to $z(t)$ satisfies (7) [5]:

$$\|T_{zw}(s)\|_\infty < \gamma, \tag{7}$$

where $\gamma > 0$ is a given performance level.

The most common algorithm for this problem is the Bisection Algorithm [14]. It is a numerical search algorithm, which simply tries to find the smallest possible γ that satisfies (7). Following is the procedure:

- Initially, lower ($\gamma_{\min}$) and upper ($\gamma_{\max}$) bounds for γ are taken as inputs
- At each iteration, average of lower and upper bounds are tried to be used as solution to $\mathcal{H}_\infty$ synthesis. This is achieved checking the associated Hamiltonian matrix of algebraic Riccati equation in 2-Riccati method and checking inequalities in LMI solution.
- If it is successful, upper bound is updated with average value; if it fails, lower bound is updated with average value.
- The algorithm stops when difference between lower and upper bounds becomes smaller than a predetermined tolerance.

2-Riccati Solution

Let a system $P(s)$ be represented as in (8).

$$P(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \tag{8}$$

Here, D_{11} and D_{22} are taken to be zero, for easing the derivations and does not cause loss of generality [5]. Following assumptions are made for the solution [2]:

(i) $(A,\ B_2)$ is stabilizable.

(ii) $(A,\ C_2)$ is detectable.

(iii) Matrix D_{12} has full column rank and the matrix D_{21} has full row rank.

(iv) Matrices P_{12} and P_{21} (2) do not have any transmission zeros located at the imaginary axis.

First two assumptions are crucial for guaranteeing the validity of admissible controllers. The third assumption involves two rank conditions on the D_{12} and D_{21} matrices. This is to ensure that there is a penalty on the control input u and the measurement y is noisy [2], [4]. The fourth assumption is to ensure admissible controllers.
Hamiltonians associated with two AREs (9) for the solution to the $\mathcal{H}_\infty$ norm calculation [2] are given in (10).

$$\begin{aligned} A^TX + XA + C_1^TC_1 - XB_2\left(D_{12}^TD_{12}\right)^{-1}B_2^TX + \frac{1}{\gamma^2}XB_1B_1^TX = 0 \\ AY + YA^T + B_1B_1^T - YC_2^T\left(D_{21}D_{21}^T\right)^{-1}C_2Y + \frac{1}{\gamma^2}YC_1^TC_1Y = 0 \end{aligned} \tag{9}$$

$$F_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix} L_\infty := \begin{bmatrix} A^T & \gamma^{-2}C_1^TC_1 - C_2^TC_2 \\ -B_1^TB_1 & -A \end{bmatrix} \tag{10}$$

The controller can be constructed using (11) [5].

$$K(s) := \begin{bmatrix} \hat{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix} \tag{11}$$

where

$$\begin{aligned} \hat{A}_\infty := A + \gamma^{-2}B_1B_1^TX_\infty + B_2F_\infty + Z_\infty L_\infty C_2 \\ F_\infty := -B_2^TX_\infty,\ L_\infty := -Y_\infty C_2^T \\ Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1} \end{aligned} \tag{12}$$

This is called the central controller (Figure 3). The collection of all admissible controllers which satisfy the condition $\|T_{zw}\|_\infty < \gamma$ is given in (13) [5].

$$M_\infty = \begin{bmatrix} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ F_\infty & 0 & I \\ -C_2 & I & 0 \end{bmatrix} \tag{13}$$

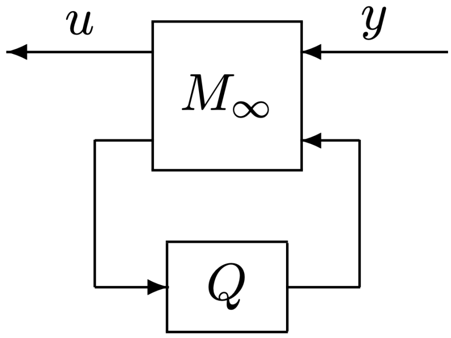

Figure 3: Central Controller and Free Parameter Q, LFT Form

where $Q \in RF_\infty$ and $\|Q\|_\infty < \gamma$. Q is a free parameter and the central controller can be obtained simply by setting $Q = 1$.

Linear Matrix Inequalities (LMI) Solution

Considering the restrictions of 2-Riccati solution, Gahinet et al. proposed a new solution to $\mathcal{H}_\infty$ synthesis problem in [3]. New solution handled the problem as a convex-optimization problem by constructing it using inequalities. Bounded Real Lemma [27] and Lyapunov Stability theorem are used in the derivation.
Considering the system represented in (8), following are equal [3]:

(i) $\|G(s)\|_\infty < \gamma$

(ii) There exists a $X > 0$ for which:
$$\begin{bmatrix} A^T X + XA & XB & C_T \\ B^T X & -\gamma I & D_T \\ C & D & -\gamma I \end{bmatrix} < 0$$

Only assumptions needed are as following [3]:

(i) (A, B_2) is stabilizable.

(ii) (A, C_2) is detectable.

(iii) $D_{22} = 0$

If these criteria hold, a dynamic output feedback controller can be synthesized as minimal realization in (14) [3].

$$K(\sigma) = D_K + C_K(\sigma I - A_K)^{-1} B_K; \quad A_K \in \mathbb{R}_{k\times k} \quad (\sigma = s, z) \tag{14}$$

The closed-loop system can be constructed using (15) [3].

$$A_{cl} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} B_{cl} = \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix} \tag{15}$$

$$C_{cl} = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix} D_{cl} = \begin{bmatrix} D_{11} + D_{12} D_K D_{21} \end{bmatrix} \tag{16}$$

With some mathematical arrangements, the system matrices are put into a form, which depend controller parameters affinely (17) [3].

$$\begin{aligned} A_{\text{cl}} &= A_C + B\,\Theta\,C, & B_{\text{cl}} &= B_C + B\,\Theta\,D_{21}, \\ C_{\text{cl}} &= C_C + D_{12}\,\Theta\,C, & D_{\text{cl}} &= D_{11} + D_{12}\,\Theta\,D_{21}. \end{aligned} \tag{17}$$

where

$$\Theta := \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \in \mathbb{R}_{(n+k)\times(n+k)} \tag{18}$$

is the controller matrix (18), and the matrices of coefficients are given in (19).

$$\begin{aligned} A_C &= \begin{pmatrix} A & 0 \\ 0 & 0_k \end{pmatrix}, \qquad B_C = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & B_2 \\ I_k & 0 \end{pmatrix}, \\ C_C &= (C_1,\, 0); \qquad C = (0,\, I_k), \\ D_{12} &= (0,\, D_{12}), \qquad D_{21} = \begin{pmatrix} 0 \\ D_{21} \end{pmatrix}. \end{aligned} \tag{19}$$

It can be seen that neither A_0, B_0 and C_0 depend on controller parameters, which makes (15) affinely depending on them.

ANALYSIS

System types used in the analyses are selected with consideration of the assumptions of the solution methods and the importance within the control theory literature. Marginally stable, unstable, non-stabilizable, non-detectable, non-minimum phase, systems with partial control authority over performance output, systems with partial observability of disturbances through measurements and systems with transmission zeros are examined.

The analyses are conducted in the MATLAB environment. The systems are modeled in Laplace domain as transfer function and in state-space domain in LFT form. The aforementioned solutions are implemented and for double check Matlab's $hinfsyn.m$, $hinfric.m$ and $hinflmi.m$ methods from from Robust Control Toolbox of Matlab [15] are used.

Marginally Stable System

Stability is widely studied in control theory. The system in hand can be already stable, e.g. a mechanical system with proper damping, or can be unstable, e.g. an inverted pendulum, or can be marginally stable, e.g. an undamped oscillator.

Undamped oscillator with band-pass characteristic is given in (20), (21), which is a marginally stable system. Note that reason for adding the band-pass characteristic to the system is creating a transmission zero located at the imaginary axis. There are two poles at the imaginary axis, $p_{1,2} = j, -j$. In (20), (21), system's transfer function and LFT representation are given.

$$G(s) = \frac{s}{s^2+1} \tag{20}$$

$$\begin{aligned} A &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \; B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \; C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ D_{11} &= \begin{bmatrix} 0 \end{bmatrix} \; D_{12} = \begin{bmatrix} 0 \end{bmatrix} \\ D_{21} &= \begin{bmatrix} 0 \end{bmatrix} \; D_{22} = \begin{bmatrix} 0 \end{bmatrix} \end{aligned} \tag{21}$$

There is a transmission zero at the imaginary axis for P_{12}. It can be observed that the system is minimum phase and marginally stable.

The open-loop system robustness can be inspected from the Bode gain and phase diagrams given in Figure 4.

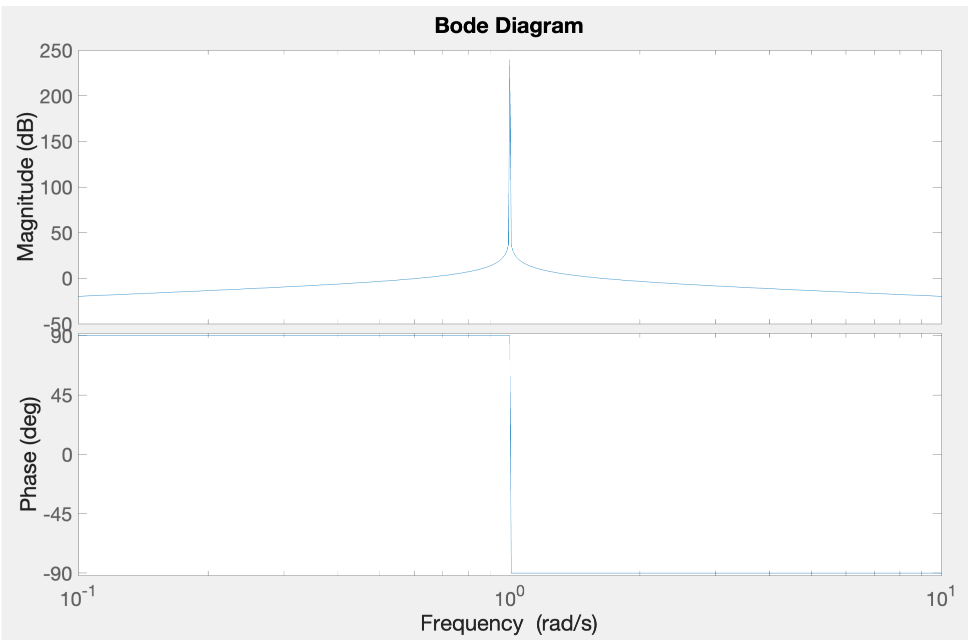

Figure 4: Bode Plot for Marginally Stable System in (20), (21)

Here, an infinite gain margin and a zero phase margin near 1 rad/sec frequency occurred, which indicate very low stability.
2-Riccati solution dictates that there should be no transmission zeros at the imaginary axis as mentioned previously. When the controller tried to be synthesized using 2-Riccati solution, result is not successful. A trick to deal with this situation is regularization, in which the matrices P_{12} and P_{21}, which dictate absence of imaginary axis transmission zeros, are perturbed to move the transmission zero away from the imaginary axis. The result when regularization option used is given in Table 1.

Iteration	γ
1	1.04 e-04
2	8.31 e-05
3	9.28 e-05
4	9.80 e-05
5	9.54 e-05
6	9.41 e-05

Table 1: γ Iteration for Marginally Stable System, using 2-Riccati Solution with Regularization

At the first glance solution seems to be converged with the γ value of $9.41e-05$. However, the controller gains (22) and the closed-loop eigenvalues (23) are not well-posed.

$$A = 1.0e{+}05 \times \begin{bmatrix} -4.8308 & 0 \\ 0 & 0 \end{bmatrix} B = 1.0e{+}05 \times \begin{bmatrix} 4.7236 \\ 0 \end{bmatrix}$$
$$C = 1.0e{+}04 \times \begin{bmatrix} -2.0976 & 0 \end{bmatrix} \; D = 0 \tag{22}$$

$$\lambda_{1,2,3,4} = 1.0e+05 \times \begin{bmatrix} -4.6162 & -0.2146 & 0 & 0 \end{bmatrix} \tag{23}$$

Large controller gains and eigenvalues indicate fast dynamics. These are not desirable, as a large controller gain is often infeasible for actuators and can easily lead to saturation of their limits. Another disadvantage is that fast dynamics tend to increase sensitivity to high frequency noises.
In order to overcome this problem, performance expectation is lowered. This is done by setting search range for γ to [1, ∞], which was [0, ∞] in previous synthesize, by default. At the end of the iterations, γ value of 1.01 is achieved (Table 2). Controller gains (24) and the closed-loop eigenvalues (25) are more proper with these settings.

Iteration	γ
1	1.01
2	1.01

Table 2: γ Iteration for Marginally Stable System, using 2-Riccati Solution with Regularization and Larger γ Lower Bound

$$A = \begin{bmatrix} -9.8871 & -1 \\ 1 & -8.0057 \end{bmatrix} B = \begin{bmatrix} 8.7603 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} -1.9968 & 0 \end{bmatrix} \; D = 0 \tag{24}$$

$$\lambda_{1,2,3,4} = \begin{bmatrix} -7.4448 & -1.7299 & -0.5781 & -0.1343 \end{bmatrix} \tag{25}$$

Following, the LMI solution is used. Nevertheless, it could not converge with γ lower bound value 0. So, γ search range is changed to $[1, \infty]$. Results is given in Table 3.

Iteration	γ
1	1.370968
2	1.059775
3	1.059775
4	1.008747
5	0.929535

Table 3: γ Iteration for Marginally Stable System, using LMI Solution

However, the synthesized controller gains (26) and the closed-loop eigenvalues (27) are not well-posed. By increasing γ lower bound, more proper controllers can be obtained, nevertheless, increasing γ means sacrificing stability in favor of performance.

$$A = \begin{bmatrix} -54.2715 & -10.9031 \\ 6.0547 & -6.2036 \end{bmatrix} \quad B = \begin{bmatrix} 15.1002 \\ 3.2957 \end{bmatrix}$$
$$C = \begin{bmatrix} -15.3365 & -1.9158 \end{bmatrix} \quad D = 0 \tag{26}$$

$$\lambda_{1,2,3,4} = \begin{bmatrix} -50.8889 & -6.2230 & -3.0851 & -0.277 \end{bmatrix} \tag{27}$$

By this example it can bee seen that due to numerical conditions, $\mathcal{H}_\infty$ synthesis for marginally stable systems is problematic for both 2-Riccati and LMI solutions.

Unstable System

For further investigation, 2-Riccati and LMI solutions are examined with an unstable system. An unstable MIMO system given in (28), (29) is modeled.

$$G = \begin{bmatrix} \frac{s+2}{s^2+s-2} & \frac{s+2}{s^2+s-2} \\ \frac{s+2}{s^2+s-2} & \frac{2s+1}{(s+2)(s^2+s-2)} \end{bmatrix} \tag{28}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \\ D_{11} &= \begin{bmatrix} 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 0 \end{bmatrix} \\ D_{21} &= \begin{bmatrix} 0 \end{bmatrix} \quad D_{22} = \begin{bmatrix} 0 \end{bmatrix} \end{aligned} \tag{29}$$

It can be seen that system poles are $p_{1,2} = -2, 1$, indicating that the system is unstable. PBH (Popov-Belevitch-Hautus) Test shows that unstable modes of the system can be stabilized, which means the system is stabilizable.

Even though stability is not an assumption of neither 2-Riccati nor LMI solutions, it is expected that unstable modes will cause numerical issues for both solutions.

When controller tried to be synthesized using 2-Riccati solution, result is not successful. Again, regularization is tried as it would be useful for numerical issues. Result when the regularization used is given in Table 4.

Iteration	γ
1	4.91 e-05
2	3.94 e-05
3	4.40 e-05
4	4.65 e-05
5	4.52 e-05

Table 4: γ Iteration for Unstable System, using 2-Riccati Solution with Regularization

Solution converged with γ value of $4.52e-05$. Nevertheless, the controller gains (30) and the closed-loop eigenvalues (31) are not proper.

$$\begin{aligned} A &= 1.0e{+}06 \times \begin{bmatrix} -1.0193 & -0.9967 \\ -0.0442 & 0 \end{bmatrix} & B &= 1.0e{+}05 \times \begin{bmatrix} 9.9666 \\ 0 \end{bmatrix} \\ C &= 1.0e{+}04 \times \begin{bmatrix} -4.4223 & 0 \end{bmatrix} & D &= 0 \end{aligned} \tag{30}$$

$$\lambda_{1,2,3,4} = 1.0e+05 \times \begin{bmatrix} -9.7403 & -0.4525 & 0 & 0 \end{bmatrix} \tag{31}$$

As in the case of the marginally stable system, changing the search range of γ can be useful. In order to give more insight into this issue, consider the equation (32) used in 2-Riccati solution.

$$A^\top X + XA + C_1^\top C_1 - XB_2R^{-1}B_2^\top X + \frac{1}{\gamma^2}XB_1B_1^\top X = 0 \tag{32}$$

It can be observed that γ term locates at the denominator and very small values of it can lead $\frac{1}{\gamma^2}XB_1B_1^\top X$ term to be very large and dominate the equation, which creates a numerical issue. Changing γ search range from [0, ∞] to [1, ∞], result is given in Table 5, a γ value of 1.01 is achieved.

Iteration	γ
1	1.01
2	1.01

Table 5: γ Iteration for Unstable System, using 2-Riccati Solution with Regularization and Larger γ Lower Bound

The controller gains (33) and the closed-loop eigenvalues (34) are more proper with this setting.

$$\begin{aligned} A &= \begin{bmatrix} -9.0042 & -5.8388 \\ -4.6181 & -2 \end{bmatrix} & B &= \begin{bmatrix} 5.8388 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} -4.6181 & 0 \end{bmatrix} & D &= 0 \end{aligned} \tag{33}$$

$$\lambda_{1,2,3,4} = \begin{bmatrix} -4.0021 + 1.394j & -4.0021 - 1.394j & -2 & -2 \end{bmatrix} \tag{34}$$

LMI solution is tried with γ search range [0, ∞]. The result is given in Table 6.

Iteration	γ	Iteration	γ
1	4.34833 e-01	12	2.02167 e-04
2	2.94807 e-01	13	9.99238 e-05
3	1.04398 e-01	14	9.99238 e-05
4	5.32610 e-02	15	7.49846 e-05
5	7.16467 e-03	16	7.49846 e-05
6	7.16467 e-03	17	5.51713 e-05
7	4.18551 e-03	18	4.94030 e-05
8	1.58925 e-03	19	4.72533 e-05
9	1.58925 e-03	20	4.48417 e-05
10	2.02167 e-04	21	4.47765 e-05
11	2.02167 e-04	22	4.47765 e-05

Table 6: γ Iteration for Unstable System, using LMI Solution

The controller gains (35) and the closed-loop eigenvalues (36) show that the synthesized controller is infeasible due to large gains.

$$A = 1.0e{+}05 \times \begin{bmatrix} -0.0011 & -1.2746 \\ -1.2745 & -2.4175 \end{bmatrix} \quad B = 1.0e{+}05 \times \begin{bmatrix} 0.0006 \\ 1.2734 \end{bmatrix}$$
$$C = 1.0e{+}05 \times \begin{bmatrix} -0.0006 & -1.2734 \end{bmatrix} \quad D = 0 \tag{35}$$

$$\lambda_{1,2,3,4} = 1.0e+05 \times \begin{bmatrix} -1.2032 + 0.3988j & -1.2032 - 0.3988j & 0 & 0 \end{bmatrix} \tag{36}$$

After updating the γ search range to [1, ∞], the result is given in Table 7. It can be observed that the solution is converged with γ value of 0.546373. The controller gains (37) and the closed-loop eigenvalues (38) are proper.

Iteration	γ
1	0.546373

Table 7: γ Iteration for Unstable System, using 2-Riccati Solution with Regularization and Larger γ Lower Bound

$$A = \begin{bmatrix} -7.0474 & -15.5278 \\ -15.5278 & -10.0313 \end{bmatrix} \quad B = \begin{bmatrix} 1.8151 \\ 13.7256 \end{bmatrix}$$
$$C = \begin{bmatrix} -1.8151 & -13.7256 \end{bmatrix} \quad D = 0 \tag{37}$$

$$\lambda_{1,2,3,4} = \begin{bmatrix} -5.1759 + 10.7098j & -5.1759 - 10.7098j & -6.7563 & -0.9706 \end{bmatrix} \tag{38}$$

Non-stabilizable Systems

Stabilizability is an important aspect of an open-loop system. An example non-stabilizable system is given in (39).

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$D_{11} = \begin{bmatrix} 0 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 1 \end{bmatrix}$$
$$D_{21} = \begin{bmatrix} 1 \end{bmatrix} \quad D_{22} = \begin{bmatrix} 0 \end{bmatrix} \tag{39}$$

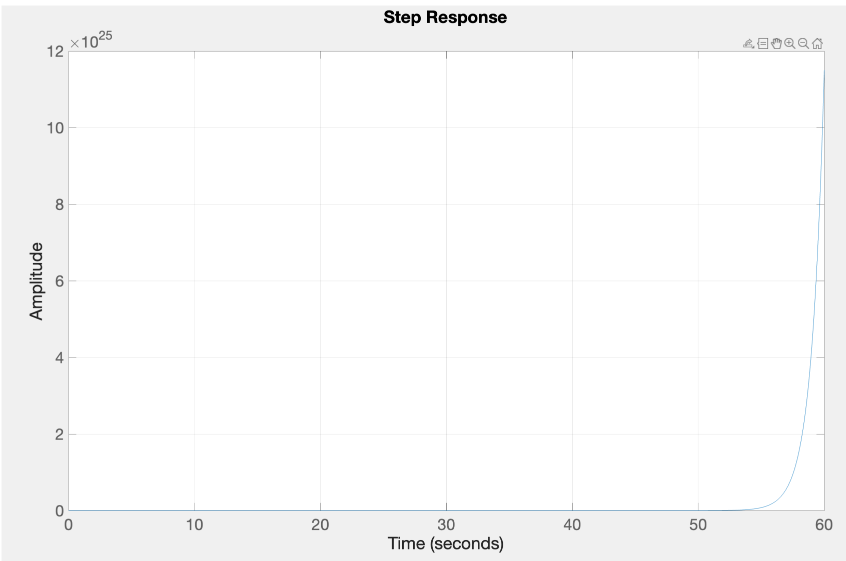

Figure 5: Step Response of the Unstable System in (39)

The system poles are $p_{1,2} = 1, -2$, indicating an unstable system, which can also be observed through open-loop step response (Figure 5).
Another property of the system in (39) is that it is non-stabilizable, which can be checked by the PBH test. The first assumption of the 2-Riccati and LMI solutions states that the system should be stabilizable. This means that no controller can be synthesized neither using 2-Riccati nor LMI solution.

Non-detectable Systems

For both 2-Riccati and LMI solutions, the first two assumptions are the same.

- $(A,\, B_2)$ is stabilizable.
- $(A,\, C_2)$ is detectable.

This assumption is needed for the existence of stabilizing dynamic output feedback controllers. Relaxing this assumption will yield to inadmissible controllers, as unstable and unobservable modes will be introduced at the imaginary axis [5].
In order to observe how does 2-Riccati and LMI solutions work with non-detectable systems, the system defined in (40) is modeled.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \; B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \\ D_{11} &= \begin{bmatrix} 0 \end{bmatrix} \qquad D_{12} = \begin{bmatrix} 1 \end{bmatrix} \\ D_{21} &= \begin{bmatrix} 1 \end{bmatrix} \qquad D_{22} = \begin{bmatrix} 0 \end{bmatrix} \end{aligned} \tag{40}$$

It can be seen that the first mode of the system is unobservable and unstable, which indicates that the system is non-detectable.
It has been observed that the 2-Riccati solution fails to synthesize a controller. Also LMI solution cannot synthesize a controller. Using regularization or larger γ lower bound are not useful as the problem is numerically ill conditioned. For 2-Riccati solution, there is no positive semi-definite stabilizing solution to observer Riccati equation.

Non-minimum Phase System

In (41), an example non-minimum phase system is given. The system represents an aircraft's pitch axis control. In some conditions, pitch control of an aircraft can exhibit non-minimum phase behavior [23].

$$\begin{aligned} A &= \begin{bmatrix} -1.623 & -0.5917 \\ 1 & 0 \end{bmatrix} \; B_1 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix} \; B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \; C_2 = \begin{bmatrix} 0.572 & -0.0549 \end{bmatrix} \\ D_{11} &= \begin{bmatrix} 0 \end{bmatrix} \qquad D_{12} = \begin{bmatrix} 0 \end{bmatrix} \\ D_{21} &= \begin{bmatrix} 0 \end{bmatrix} \qquad D_{22} = \begin{bmatrix} 0 \end{bmatrix} \end{aligned} \tag{41}$$

The system poles are $p_{1,2} = -1.07, -0.553$ and the system zeros are $z_{1,2,3,4} = 0, 2.9585, 7.3703, 0.096$. Transmission zeros for P_{12} and P_{21} are given in (42).

$$P_{12_{tz}} = \begin{bmatrix} 0 \end{bmatrix} \; P_{21_{tz}} = \begin{bmatrix} 7.3703 \end{bmatrix} \tag{42}$$

It can be observed that system is non-minimum phase, stable and has transmission zero at the imaginary axis. Figure 6 shows the non-minimum phase characteristic.

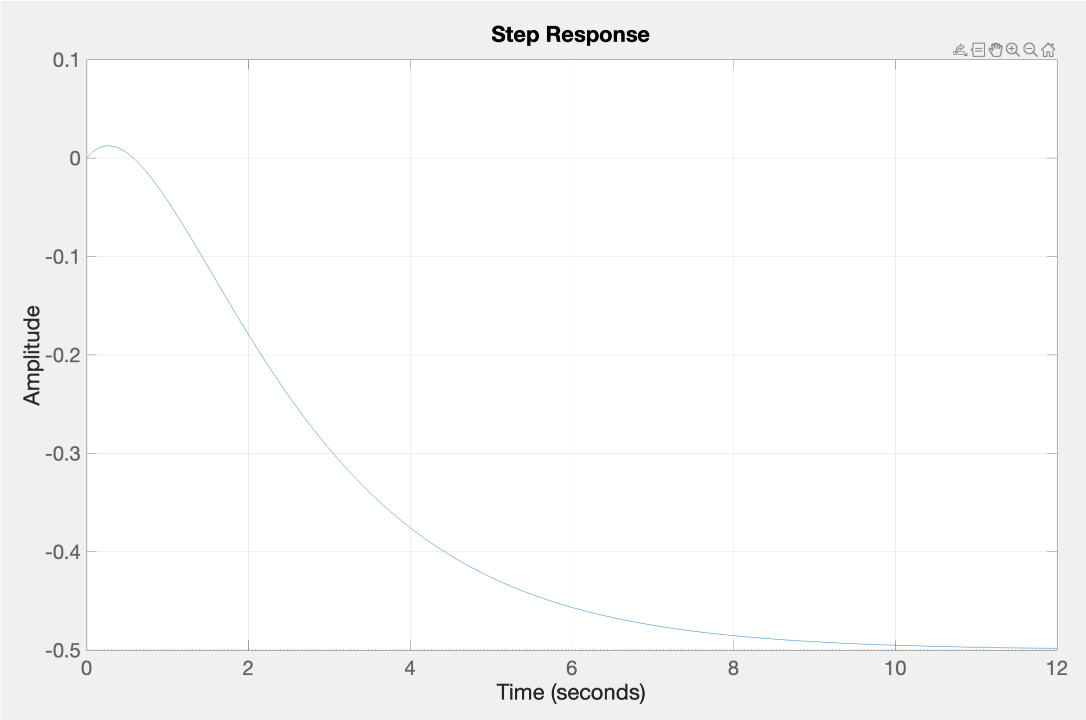

Figure 6: Step Response of the Non-minimum Phase System in (41)

It is not expected from 2-Riccati solution to synthesize an admissible controller due to imaginary axis zeros. Even though regularization and larger values of γ lower bound is used, no $\mathcal{H}_\infty$ controller could be synthesized with 2-Riccati solution.
For LMI solution γ lower bound is kept larger and regularization is used. The result is in Table 8, γ value of 0.972483 is achieved. The controller gains (43) and the closed-loop eigenvalues (44) are large, but can be said to be proper.

$$A = \begin{bmatrix} -33.469 & -5.0387 \\ -17.6915 & -6.8712 \end{bmatrix} \quad B = \begin{bmatrix} 7.7302 \\ 9.4376 \end{bmatrix}$$
$$C = \begin{bmatrix} -5.3756 & 0.1475 \end{bmatrix} \qquad D = 0 \tag{43}$$

$$\lambda_{1,2,3,4} = \begin{bmatrix} -35.7246 & -4.9106 & -0.664 + 0.1817j & -0.664 - 0.1817j \end{bmatrix} \tag{44}$$

Iteration	γ
1	2.096751
2	1.083032
3	1.053457
4	1.009868
5	0.972493

Table 8: γ Iteration for Non-minimum Phase System, using LMI Solution with Larger γ Lower Bound

Control Authority Over Performance Output

To be able to influence the performance output, the control input should be able to influence it directly. This should be handled in the problem formulation step. For example, if the object is the active suspension control of a car, the vertical acceleration should able to be controlled with the actuator's generated force.

In a mathematical sense, this corresponds to whether the D_{12} matrix has full column rank or not. To investigate this, a simple system is modeled (45).

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \; B_1 = \begin{bmatrix} -0.7 \\ 0.45 \end{bmatrix} \; B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1.2 & -0.8 \end{bmatrix} \; C_2 = \begin{bmatrix} -0.35 & 0.9 \end{bmatrix} \\ D_{11} &= \begin{bmatrix} 0 \end{bmatrix} \; D_{12} = \begin{bmatrix} 0 \end{bmatrix} \\ D_{21} &= \begin{bmatrix} 1 \end{bmatrix} \; D_{22} = \begin{bmatrix} 1 \end{bmatrix} \end{aligned} \tag{45}$$

It can be seen that D_{12} does not have full column rank, as it equals to *0*.
When the controller tried to be synthesized using the 2-Riccati solution, result is not successful. This is expected as the rank condition on D_{12} is violated. To overcome this issue, regularization is applied. In (46) D before and after the regularization is given.

$$D_{\text{before-regularization}} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \; D_{\text{after-regularization}} = \begin{bmatrix} 0 & -0.0006325 \\ 1 & 1 \end{bmatrix} \tag{46}$$

Even though the applied regularization is very small, synthesis is successful in this case. Result of the iteration process is given in Table 9, γ value of $3.73e-01$ is achieved.

Iteration	γ	**Iteration**	γ
1	5.30 e-03	8	3.84 e-01
2	6.26 e-02	9	3.77 e-01
3	2.15 e-01	10	3.73 e-01
4	3.99 e-01	11	3.73 e-01
5	2.93 e-01	12	3.73 e-01
6	3.42 e-01	13	3.73 e-01
7	3.69 e-01		

Table 9: γ Iteration for System with not Full Column Rank D_{12} matrix, using 2-Riccati Solution with Regularization

When the LMI solution is used for the synthesis, no regularization is needed, and the achieved γ value is very similar to the one achieved with the 2-Riccati solution. Result is given in Table 10, γ value of 0.371159 is achieved.

Iteration	γ	**Iteration**	γ
1	1.774668	10	0.408630
2	1.164654	11	0.392929
3	0.948926	12	0.392929
4	0.577085	13	0.372414
5	0.481079	14	0.372414
6	0.481079	15	0.372414
7	0.453450	16	0.371530
8	0.424354	17	0.371530
9	0.424354	18	0.371159

Table 10: γ Iteration for System with not Full Column Rank D_{12} matrix, using LMI Solution

Direct Observability of Disturbances Through Measurements

Disturbances should be observable in order to be compensated. Direct observation of disturbances is not generally possible, so their effect on the system measurement should be inspected. If effects of the

disturbances cannot be observed through the system measurements, there can be no compensation for them.
D_{21} is the decisive matrix for the observation of disturbances. As depicted in the assumptions, D_{21} should have full row rank.
To investigate this simple system in (47) is modeled.

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \; B_1 = \begin{bmatrix} -0.7 \\ 0.45 \end{bmatrix} \; B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1.2 & -0.8 \end{bmatrix} \; C_2 = \begin{bmatrix} -0.35 & 0.9 \end{bmatrix} \\ D_{11} &= \begin{bmatrix} 0 \end{bmatrix} \; D_{12} = \begin{bmatrix} 1 \end{bmatrix} \\ D_{21} &= \begin{bmatrix} 0 \end{bmatrix} \; D_{22} = \begin{bmatrix} 1 \end{bmatrix} \end{aligned} \tag{47}$$

It can be seen that D_{21} does not have full row rank as it equals to *0*.
When controller tried to be synthesized using 2-Riccati solution, result is not successful. This is expected as the rank condition on D_{12} is violated. To overcome this issue, regularization is applied. In (48), $D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ matrix before and after the regularization is given.

$$D_{before-regularization} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \; D_{after-regularization} = \begin{bmatrix} 0 & 1.882e-05 \\ 1 & 1 \end{bmatrix} \tag{48}$$

Even though the applied regularization is very small, the synthesis is successful in this case. Result of the iteration process is given in Table 11, γ value of $4.43e+05$ is achieved.

Iteration	γ
1	4.87 e-05
2	4.57 e-05
3	4.43 e-05

Table 11: γ Iteration for System with not Full Column Rank D_{21} matrix, using 2-Riccati Solution with Regularization

When the LMI solution is used for the synthesis, no regularization is needed and the solution is converged. Result is given in Table 12, γ value of $5.36720e-05$ is achieved.

Iteration	γ	**Iteration**	γ
1	3.4724 e-01	13	1.8431 e-04
2	2.3010 e-01	14	1.8431 e-04
3	8.8398 e-02	15	7.7948 e-05
4	4.6475 e-02	16	7.7948 e-05
5	7.3034 e-03	17	7.1577 e-05
6	7.3034 e-03	18	6.2584 e-05
7	2.7252 e-03	19	5.7389 e-05
8	2.7252 e-03	20	5.5258 e-05
9	5.9790 e-04	21	5.4569 e-05
10	5.9790 e-04	22	5.4193 e-05
11	2.5986 e-04	23	5.3872 e-05
12	2.5986 e-04	24	5.3672 e-05

Table 12: γ Iteration for System with not Full Column Rank D_{21} matrix, using LMI Solution

Plants with Transmission Zeros at the Imaginary Axis

Some of the previous systems had transmission zeros at imaginary axis, and its effect on the 2-Riccati solution is studied. It has been observed that using regularization, the issue can be handled.

Nevertheless, in some cases γ-iteration algorithm encounters some discontinuities. This generally happens for the systems, which are augmented with pre-compensators/filters in mixed sensitivity approach. When systems have zeros, which are close to the imaginary axis, in some cases it causes to the pole/zero cancellations. The example here will be a general one, which has transmission zero at the imaginary axis, the outcome should be considered for the aforementioned systems, especially the ones augmented in the mixed sensitivity approach.
In (49), a system with a transmission zero at the imaginary axis is given. The example is worked on [3].

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \end{bmatrix} \; C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D_{11} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \; D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
D_{21} &= \begin{bmatrix} 1 \end{bmatrix} \; D_{22} = \begin{bmatrix} 0 \end{bmatrix}
\end{aligned} \tag{49}
$$

P_{12} has a transmission zero at the imaginary axis. In order to observe the discontinuity for γ, the system matrix will be perturbed using (50).

$$A = A + \epsilon I \tag{50}$$

To show the discontinuity with the 2-Riccati solution, ϵ values of $\pm\ 1.0e-04$, $\pm\ 1.0e-06$, and $\pm\ 1.0e-08$ are used. Regularization is used for pushing the transmission zero at the imaginary axis away. Result is given in Table 13.

ϵ	γ for A-ϵI	γ for A+ϵI
1.0 e-04	0.9	2
1.0 e-06	0.9	2
1.0 e-08	0.9	0.9

Table 13: Discontinuity of γ due to Perturbation, using 2-Riccati Solution

The discontinuity can be seen as the perturbation gets closer to the zero, the achieved γ suddenly jumps from 2 to 0.9.
For the LMI solution, regularization is not used as it is not necessary. Result is given in Table 14.

ϵ	γ for A-ϵI	γ for A+ϵI
1.0 e-04	2	0.9
1.0 e-06	2	0.9
1.0 e-08	2	0.9

Table 14: No Discontinuity of γ due to Perturbation, using LMI Solution

It can be seen that the previous discontinuity for ϵ value $1.0e-08$ is solved. Nevertheless, the difference between γ values for positive and negative perturbations is problematic. The problem is caused by large values of variable S in LMI equations. By penalizing the large values of R and S, the result in Table 15 is obtained. It can be seen that the discontinuity no longer exists.

ϵ	γ for A-ϵI	γ for A+ϵI
1.0 e-04	2	2
1.0 e-06	2	2
1.0 e-08	2	2

Table 15: No Discontinuity of γ due to Perturbation, using LMI Solution

RESULTS AND EVALUATIONS

Throughout this paper, $\mathcal{H}_\infty$ controllers are synthesized for different system types using both 2-Riccati and LMI solutions.
For marginally stable system, regularization is used for transmission zero located at the imaginary axis. Nevertheless, synthesized controller is not proper. In order to synthesize a proper controller, lower bound for γ is increased. Same controller feasibility issue occurred in LMI solution as well, and increasing lower bound for gamma solved the issue.
For unstable system, same procedure is applied for both solutions and the achieved γ values are very similar. This can be interpreted as for proper systems, both solution work similar and well.
Non-stabilizable system is a hard problem for $\mathcal{H}_\infty$ synthesis as stabilizability is the first condition for the solution. In this case, neither of the solutions could not synthesize an admissible controller.
Non-detectable system is a very good example for $\mathcal{H}_\infty$ synthesis as real-world applications generally fall into this category. Again, neither of the solutions yield a result.
For non-minimum phase system, the 2-Riccati solution is unable to synthesize a controller due to inherent imaginary axis zero. Nevertheless, the LMI solution synthesized an admissible controller.
By checking D_{12} column rank condition, effect of the control authority over the performance output is investigated. It has been observed that using regularization for 2-Riccati solution, achieved result is very similar to the one in LMI solution.
Similarly, by checking D_{21} row rank condition, effect of the direct observability of disturbances through measurements is investigated. It has been observed that using regularization for the 2-Riccati solution, achieved result is very similar to the LMI solutions.
The first conclusion is that for proper systems, which fulfill assumptions of 2-Riccati and LMI solutions, the performance of synthesized controllers are the same. In these cases, 2-Riccati solution can be chosen over the LMI solution, concerning computational complexity.
Through different system types, it is a general observation that the 2-Riccati solution is more sensitive to the plant variations than the LMI solution. This is especially problematic for systems with small gain and phase margins. In these cases choosing LMI solution over the 2-Riccati solution is recommended.
Another important learning is that the problem setup is very crucial to $\mathcal{H}_\infty$ synthesis for both solutions. As solutions assumptions pose serious restrictions on the system, unimportant states and parameters should be left out of the model.
Through the studies which take part in this paper, an important conclusion is that the LMI solution possesses more tricks than the 2-Riccati solution for dealing with numerical issues arising from modeling or problem settings.
As learned from the plant with transmission zero at the imaginary axis, even when no assumption is violated for the LMI solution, discontinuities can still occur. For that reason design and analysis should be done carefully considering the regularization, the penalization and the normalization of the variables of the inequalities.

CONCLUSION

In this paper, 2-Riccati and LMI solutions to the $\mathcal{H}_\infty$ synthesis are investigated across different system types. Marginally stable, unstable, non-stabilizable, non-detectable, non-minimum phase, control authority over performance output, direct observability of disturbances through measurements, and plants with transmission zeros at the imaginary axis systems and behaviors are examined. In analysis, iterations through bisection algorithm and intermediate steps are shown. Synthesized controllers are examined regarding their feasbility.

It has been observed that both 2-Riccati and LMI solutions are sensitive to the numerical conditions. Nevertheless, LMI solution deals better with them. Another important conclusion is that synthesized $\mathcal{H}_\infty$ controller should be checked whether if the gains are feasible or not. Even in the cases where the synthesized controller seems to be proper, discontinuities may still occur.

As a future study, feasibility of the synthesized controllers and the cases, where synthesis fails can be investigated in a mathematical sense. Also, maximum entropy solution can be included to the comparisons.

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